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<u>Chirality of two-fluid magnetohydrodynamics</u>

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ERC Advanced Grant (N. 339032) "NewTURB"
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Chirality of two-fluid magnetohydrodynamics

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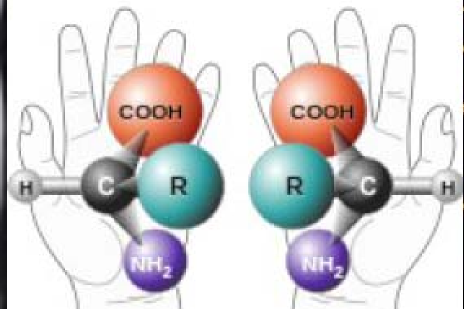
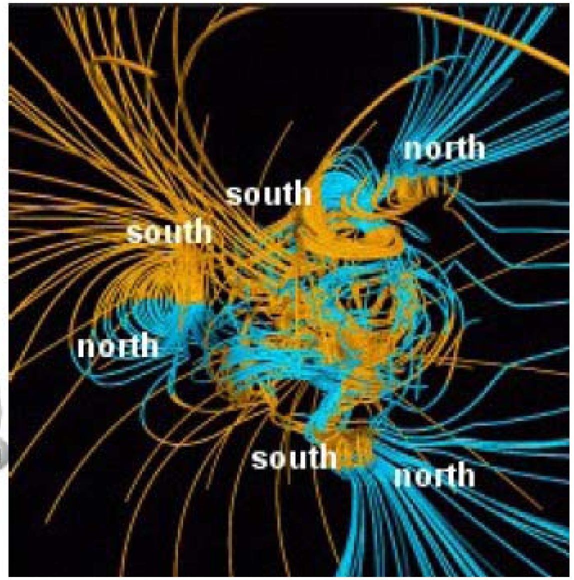
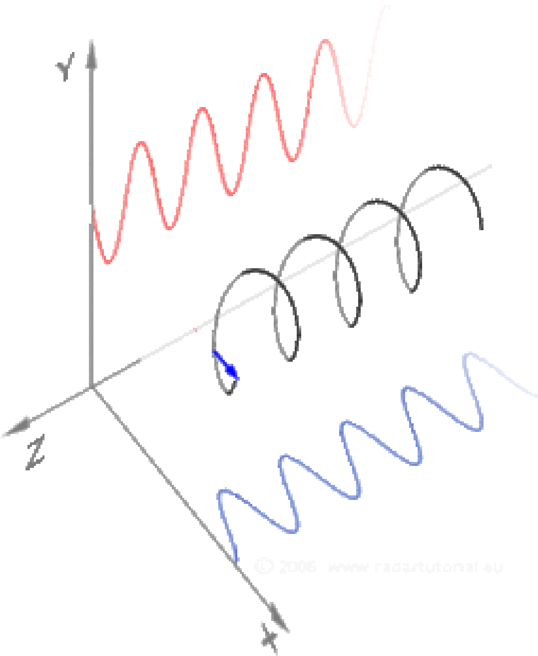
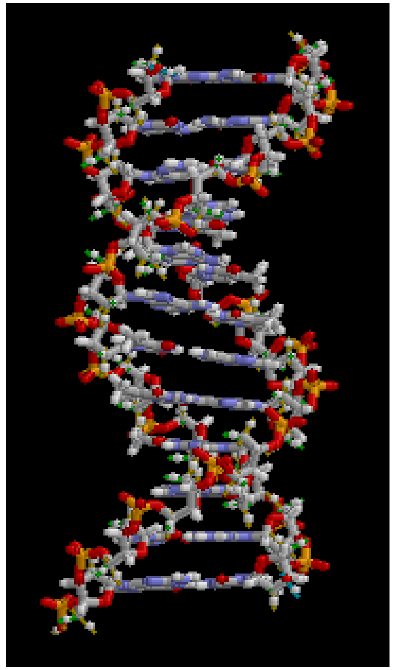




Somewhat interdisciplinary ‘References’ or history of my own relevant to helicity

- Jian-Zhou Zhu et al., 2008 (unsubmitted) on **helical** Navier-Stokes and EDQNM;
- Jian-Zhou Zhu, 2012 (unpublished) on two-dimensional gyrokinetics;
- Jian-Zhou Zhu, Weihong Yang and Guang-Yu Zhu 2014 published in *Journal of Fluid Mechanics*, “Purely helical absolute equilibria and chirality of (magneto)fluid turbulence”;
- Jian-Zhou Zhu, 2014 published in *Physics of Fluids*, “Note on specific chiral ensembles of statistical hydrodynamics: ‘Order function’ for transition of turbulence transfer scenarios”;
- Jian-Zhou Zhu, 2014 “Exact solutions of some hydrodynamic type models: restricted superposition of helical waves” (soon to appear on *arXiv.org*);
- Jian-Zhou Zhu, 2014 “Is inverse cascade of passive scalar energy advected by incompressible two-dimensional turbulence genuinely possible?” (soon to appear on *arXiv.org*);
- Jian-Zhou Zhu et al. 2014 “On chirality of two-fluid magnetohydrodynamic turbulence” (to appear on *arXiv.org*);
- Jian-Zhou Zhu et al. 2014 “Disordered DNA Neutral Evolution: Algebraic Tails of Self-Alignment Concentrations” (to appear on *arXiv.org*).





from wikipedia.org

Outline

- Partially and fully two-fluid MHDs (tfMHD) for plasmas
- Helical waves in tfMHD and helical representation: “novel” derivation of the dispersion relation and exact nonlinear (wave) solutions
- Chiralities in the sense of Meyrand and Galtier, and, in the sense of helical modes: Detailed eMHD-iMHD analyses and unification
- Conclusion

Partially and fully two-fluid MHDs (tfMHD)

Ideal Hall MHD equation (dimensionless):

$$\frac{\partial \boldsymbol{v}}{\partial t} = -(\boldsymbol{v} \cdot \nabla)\boldsymbol{v} + (\boldsymbol{b} \cdot \nabla)\boldsymbol{b} - \nabla P \quad (1)$$

$$\frac{\partial \boldsymbol{b}}{\partial t} = (\boldsymbol{b} \cdot \nabla)\boldsymbol{v} - (\boldsymbol{v} \cdot \nabla)\boldsymbol{b} \underline{- \epsilon \nabla \times [(\nabla \times \boldsymbol{b}) \times \boldsymbol{b}]} \quad (2)$$

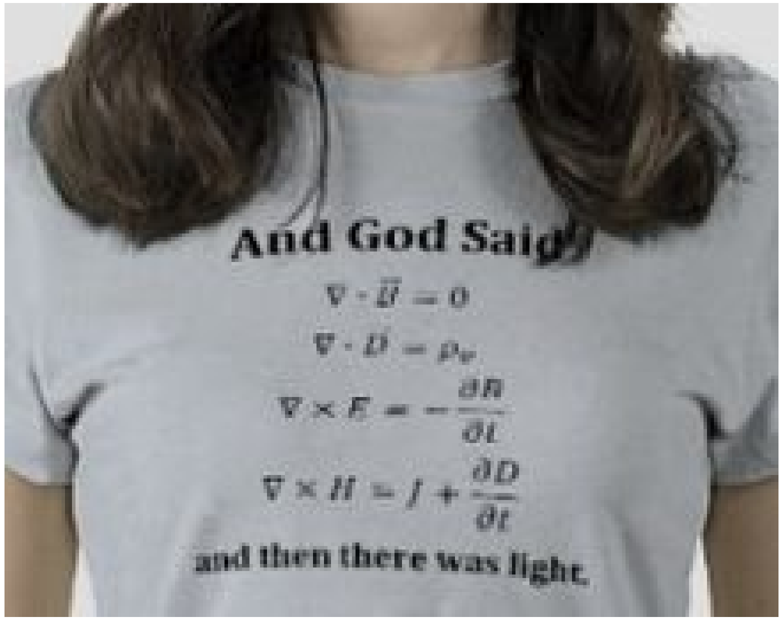
$$\nabla \cdot \boldsymbol{v} = 0 \quad (3)$$

$$\nabla \cdot \boldsymbol{b} = 0 \quad (4)$$

Reminding Navier-Stokes

$$\frac{dv}{dt} = \frac{\partial v}{\partial t} + \nabla \frac{v^2}{2} + \Omega \times v = F - \frac{1}{\rho}\nabla p + \nu \Delta v + \frac{1}{3}\nu \nabla (\nabla \cdot v)$$

Helmholtz, Kelvin



$$\frac{\partial \Omega}{\partial t} + \nabla \times (\Omega \times v)$$

$$= \nabla \times F - \nabla \times \left(\frac{1}{\rho}\nabla p\right) + \nabla \times (\nu \Delta v) + \frac{1}{3}\nabla \times (\nu \nabla (\nabla \cdot v))$$

Frozen-in form

Hall ("partially" two-fluid):

$$\partial_t \boldsymbol{\Omega}_j = \nabla \times (\boldsymbol{v}_j \times \boldsymbol{\Omega}_j), \qquad (j = R, L)$$

$$\boldsymbol{\Omega}_R = \boldsymbol{b}, \boldsymbol{v}_R = \boldsymbol{v} - \epsilon \nabla \times \boldsymbol{b} \text{ and } \boldsymbol{\Omega}_L = \boldsymbol{b} + \epsilon \nabla \times \boldsymbol{v}, \boldsymbol{v}_L = \boldsymbol{v}$$

(MHD is not recovered by simply letting \epsilon be zero here;
the two vortexes nonlinearly couple by sharing the same **b** and **v**.)

"partially": j =//= i or e

Fully two-fluid:

$$\partial_t \boldsymbol{\Omega_j} = \nabla \times (\boldsymbol{v_j} \times \boldsymbol{\Omega_j}), \qquad (j = i, e)$$

$$\boldsymbol{\Omega}_j = \nabla \times \boldsymbol{P}_j = m_j \boldsymbol{w}_j + q_j \boldsymbol{b},\ \boldsymbol{w}_j = \nabla \times \boldsymbol{v}_j,\ \boldsymbol{P}_j = m_j \boldsymbol{v}_j + q_j \boldsymbol{a}$$

(two “Navier-Stokes”!)

The nonlinear coupling between two vortexes is through sharing the same **b**.

Meyrand-Galtier’s hMHD chirality (very interesting and beautiful, but not necessarily correct)

Spontaneous Chiral Symmetry Breaking of Hall Magnetohydrodynamic Turbulence

Chirality and polarization.—We define the normalized magnetic helicity and cross correlation as, respectively:

$$\sigma_m = \frac{\hat{\mathbf{a}} \cdot \hat{\mathbf{b}}^* + \hat{\mathbf{a}}^* \cdot \hat{\mathbf{b}}}{2|\hat{\mathbf{a}}||\hat{\mathbf{b}}|}, \qquad \sigma_c = \frac{\hat{\mathbf{u}} \cdot \hat{\mathbf{b}}^* + \hat{\mathbf{u}}^* \cdot \hat{\mathbf{b}}}{2|\hat{\mathbf{u}}||\hat{\mathbf{b}}|},$$

where $\hat{}$ means the Fourier transform, * the complex conjugate, and **a** the magnetic vector potential. From these quantities, we may define the magnetic polarization, $P_m = \sigma_m \sigma_c$, which varies by definition between -1 and $+1$. Hall MHD supports R and L circularly polarized waves for which we have respectively $P_m = -1$ and $+1$ [6]; the first case corresponds to incompressible whistler waves (also called kinetic Alfvén waves [18]) and the second to ion-cyclotron waves. By extension, in our numerical study we define the R and L fluctuations for which we have, respectively, $P_m < 0$ and $P_m > 0$. Note that the forcing terms in Eqs. (1) and (2) are chosen such as injection rates of cross helicity and magnetic helicity are null.

Numerical result and theoretical interpretation

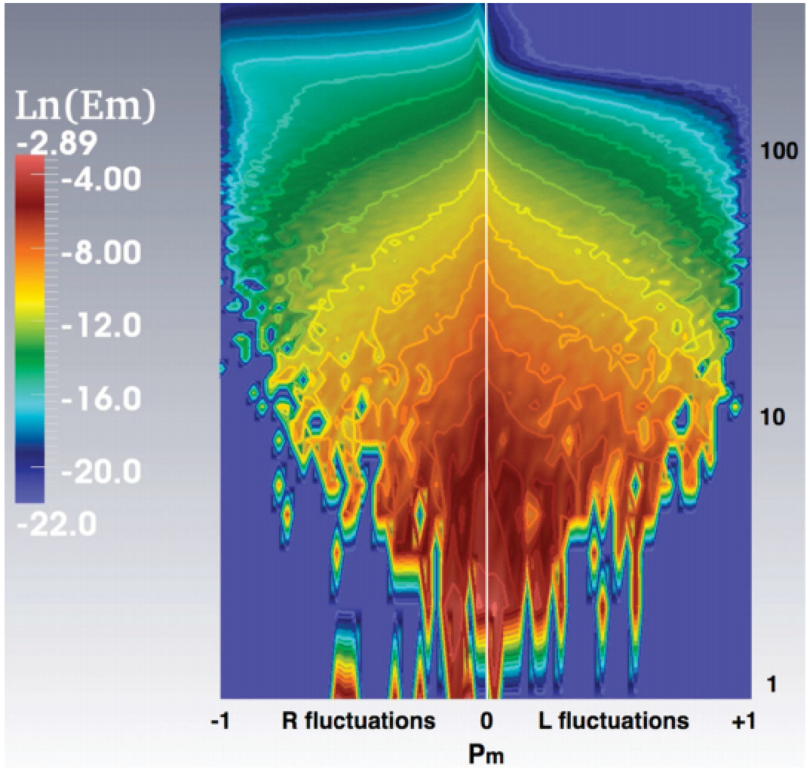


FIG. 1 (color). Magnetic energy as a function of P_m and k (in logarithmic coordinate). Isocontours of energy (in logarithmic scale) are displayed in order to separate the regions of high energy (red) from those of low energy (blue).

Theoretical interpretation.—A simple theoretical interpretation of our numerical simulations may be given by rewriting the Hall MHD equations as follows [19]:

$$\partial_t \mathbf{\Omega_j} = \nabla \times (\mathbf{u}_j \times \mathbf{\Omega_j}), \qquad (j = R, L), \qquad (5)$$

with the pair of generalized vortices and velocities ($\mathbf{\Omega}_R = \mathbf{b}$, $\mathbf{u}_R = \mathbf{u} - d_I \nabla \times \mathbf{b}$) and ($\mathbf{\Omega}_L = \mathbf{b} + d_I \nabla \times \mathbf{u}$, $\mathbf{u}_L = \mathbf{u}$). We first note that the generalized vorticities $\Omega_{R,L}$ are frozen in the flow $\mathbf{u}_{R,L}$. Let us imagine a turbulent flow in which just one type of generalized vortex evolves, say vortices $\mathbf{\Omega}_R$. In this particular regime $\mathbf{u}_L$ must be equal to zero; then we recover the well known EMHD regime described above. Now let us imagine that only $\mathbf{\Omega}_L$ vortices evolve: in this case, we must have $\mathbf{u}_R = 0$ which implies the condition $\mathbf{u} = d_I \nabla \times \mathbf{b}$. Under these conditions, Eq. (5) becomes:

$$\partial_t (1 - d_I^2 \triangle)\mathbf{b} = d_I \nabla \times [(\nabla \times \mathbf{b}) \times (1 - d_I^2 \triangle)\mathbf{b}]. \qquad (6)$$

Linearising Eq. (6) about a static homogeneous magnetic field $\mathbf{B}_0$ gives $\omega_L (1 + d_I^2 k^2)\hat{\mathbf{b}} = d_I k_\parallel B_0 i\mathbf{k} \times \hat{\mathbf{b}}$, which yields in the limit $kd_I \gg 1$ nothing else than the dispersion relation of the left-handed circularly polarized cyclotron waves, i.e., $\omega_L = B_0 k_\parallel / (kd_I)$ [20]. How can we interpret

chirality in the sense of helical representation
(selection and amplification in the absolute equilibrium, polarized linear wave, nonlinear unichiral solution)

Helical representation

Then, for a 3D transverse vector field $\boldsymbol{v}$, the helical mode/wave representation in Fourier space reads[4,7–9]

$$\boldsymbol{v} = \sum_c \boldsymbol{v}^c = \sum_{\boldsymbol{k},c} \hat{\boldsymbol{v}}^c(\boldsymbol{k}) e^{\hat{i}\boldsymbol{k}\cdot\boldsymbol{r}} = \sum_{\boldsymbol{k},c} \hat{v}^c(\boldsymbol{k}) \hat{\boldsymbol{h}}_c(\boldsymbol{k}) e^{\hat{i}\boldsymbol{k}\cdot\boldsymbol{r}}. \quad (1)$$

Here $\hat{i}^2 = -1$ and $c^2 = 1$ for the chirality indexes $c =$ "+" or "−". [For consistency of notation, every complex variable wears a hat and its complex conjugate is indexed by "*".] For convenience, we normalize the box to be of 2π period and that $k \geq 1$. The helical mode bases (complex eigenvectors of the curl operator) have the following properties:

$$\hat{i}\boldsymbol{k} \times \hat{\boldsymbol{h}}_c(\boldsymbol{k}) = ck\hat{\boldsymbol{h}}_c(\boldsymbol{k}),$$

$$\hat{\boldsymbol{h}}_c(-\boldsymbol{k}) = \hat{\boldsymbol{h}}_c^*(\boldsymbol{k}) = \hat{\boldsymbol{h}}_{-c}(\boldsymbol{k})$$

and $\hat{\boldsymbol{h}}_{c_1}(\boldsymbol{k}) \cdot \hat{\boldsymbol{h}}_{c_2}^*(\boldsymbol{k}) = \delta_{c_1,c_2}$ (Euclidean norm). $\hat{\boldsymbol{h}}_c(\boldsymbol{k}) e^{\hat{i}\boldsymbol{k}\cdot\boldsymbol{r}}$ is the eigenfunction of the curl operator corresponding to the eigenvalue ck. Or, with the case $c = 0$ also included for the compressible field, the variable c "itself may be considered to be the eigenvalue of the operator $(-\nabla^2)^{-1/2}\nabla\times$ when this operator is properly interpreted."[7] The bases can be simply constructed as[4,10]

$$\hat{\boldsymbol{h}}_c(\boldsymbol{k}) = (c\hat{i}\boldsymbol{p} + \boldsymbol{p} \times \boldsymbol{k}/k)/(\sqrt{2}p),$$

with $\boldsymbol{p}$ being perpendicular to $\boldsymbol{k}$. The structure $\hat{\boldsymbol{h}}_c(\boldsymbol{k}) e^{\hat{i}\boldsymbol{k}\cdot\boldsymbol{r}}$ is common in inertial waves of rotating fluids and cyclotron waves of plasmas, being circularly polarized, with $c = \pm$ representing opposite chirality. For better or alternative physical intuition, we may conveniently call

$$\breve{\boldsymbol{v}}^c(\boldsymbol{r}|\boldsymbol{k}) = \hat{v}^c(\boldsymbol{k}) \hat{\boldsymbol{h}}_c(\boldsymbol{k}) e^{\hat{i}\boldsymbol{k}\cdot\boldsymbol{r}} + c.c.,$$

with $c.c.$ for "complex conjugate", a "chiroid" which is maximally/purely helical or of highest degree of chriality, since the helicity contribution of it is

$$\nabla \times \breve{\boldsymbol{v}}^c(\boldsymbol{r}|\boldsymbol{k}) \cdot \breve{\boldsymbol{v}}^c(\boldsymbol{r}|\boldsymbol{k}) = 2ck|\hat{v}^c(\boldsymbol{k})|^2 = ck|\breve{\boldsymbol{v}}^c(\boldsymbol{r}|\boldsymbol{k})|^2;$$

etc14

HALL MAGNETOHYDRODYNAMIC HARMONIC-HELICON ABSOLUTE EQUILIBRIUM

single-fluid MHD and electron MHD. Here, I present the Hall MHD results. This model is Hamiltonian with the canonical momenta (see, e.g., [14, 15]) $\boldsymbol{p}_i = m_e \boldsymbol{u} + q_i \boldsymbol{A}$ and $\boldsymbol{p}_e = q_e \boldsymbol{A}$, from which one can find the rugged invariants, magnetic helicity $\mathcal{H}_M = \frac{1}{2\mathcal{V}} \int \boldsymbol{A} \cdot \boldsymbol{B} d^3\boldsymbol{r}$ and "generalized" helicity $\mathcal{H}_G = \frac{1}{2\mathcal{V}} \int (\boldsymbol{u} \cdot \boldsymbol{B} + \frac{\epsilon}{2} \boldsymbol{\omega} \cdot \boldsymbol{v}) d^3\boldsymbol{r}$, besides total energy $\mathcal{E} = \frac{1}{2\mathcal{V}} \int [\boldsymbol{u}^2 + \boldsymbol{B}^2] d^3\boldsymbol{r}$, and the spectral densities are:

$$U_K^c(k) = -4 \frac{\alpha\, k + c\beta_M}{D_H^c},\ U_M^c(k) = -2 \frac{(2\,\alpha + c\beta_G\, \epsilon\, k)\, k}{D_H^c},\ Q_M^c(k) = \frac{c}{k} U_M^c(k),\ \text{and } Q_G^c(k) = 2 \frac{\beta_G\, k}{D_H^c} + c\frac{\epsilon}{2} k U_K^c(k),$$

with $D_H^c(k) = -c \cdot 2\alpha\, \beta_G\, \epsilon\, k^2 - (4\,\alpha^2 + 2\,\beta_G\, \epsilon\, \beta_M + {\beta_G}^2)k - c \cdot 4\,\alpha\, \beta_M$. The new notations follow the rule in the above and are explained by themselves. Summation over the c index produces Servidio et al. [16]: For comparison, I have used exactly the same form of invariants as theirs and that my α corresponds to their β, β_G to their γ and β_M to their θ.

The poles of opposite chiral sectors have opposite signs.
pole → one chiral sector dominated states (OCSDS)

Sidenote for Navier-Stokes: $U_K^c(k) = 1/(\alpha + c\beta k),\quad Q_K^c(k) = ckU_K^c(k)$

(circularly) polarized waves

equation). We use $\boldsymbol{U}_s = \boldsymbol{u}_s, \boldsymbol{B} = \boldsymbol{B}_0 + \boldsymbol{b}$ with $\nabla \times \boldsymbol{B}_0 = 0$. The two-fluid "frozen-in" equations then become

$$\partial_t(m_s \nabla \times \boldsymbol{u}_s + q_s \boldsymbol{b}) = \nabla \times [\boldsymbol{u}_s \times (m_s \nabla \times \boldsymbol{u}_s + q_s \boldsymbol{b})] + q_s \nabla \times (\boldsymbol{u}_s \times \boldsymbol{B}_0) \quad (20)$$

which, as said, is solved by the Beltrami wave (to which we are limiting ourselves for the time being), Eq. (6), i.e.,

$$\boldsymbol{u}_s = [\sum_{|\boldsymbol{k}|=k_s, c=c_s^*} \hat{u}_s^c(\boldsymbol{k}) \hat{\boldsymbol{h}}_c(\boldsymbol{k}) e^{\hat{i}\boldsymbol{k}\cdot\boldsymbol{r}}] e^{-\hat{i}\omega t},$$

with c_s^* being uniformly $+$ or $-$ and k_s constants. The dispersion relation can be obtained in the conventional way (Stix 1992) or more straightforwardly by using helical representation from the beginning as follows, which will also settle down the relations among k_s and α_s with $\boldsymbol{b} = \alpha_s \boldsymbol{u}_s$. The dispersion relation is determined by the Maxwell equations and the linear part of Eq. (20), that is

$$\partial_t(m_s \nabla \times \boldsymbol{u}_s + q_s \boldsymbol{b}) = q_s \nabla \times (\boldsymbol{u}_s \times \boldsymbol{B}_0). \quad (21)$$

From the above helical representation, as in the conventional derivation of dispersion relation with mono-wavelength, but now also uni-chiral wave, we replace in the above equation $\boldsymbol{u}_s$ with Eq. (14), i.e.,

$$\hat{\boldsymbol{u}}_s^c = \hat{u}_s^c(\boldsymbol{k}) \hat{\boldsymbol{h}}^c(\boldsymbol{k}) e^{\hat{i}(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)},$$

and similarly for $\boldsymbol{b}$, which from the above linear equation (21) and $\nabla \times \boldsymbol{b} = \mu_0 \sum_s q_s n_s \boldsymbol{u}_s \cancel{+\partial_t \boldsymbol{E}}$ (for simplicity, we have neglected the displacement current $\partial_t \boldsymbol{E}$ which of course could be included in the calculation for more general results), leads to:

$$\omega[c m_s k \hat{u}_s^c(\boldsymbol{k}) + q_s \hat{b}^c(\boldsymbol{k})] = -q_s B_0 k_\parallel \hat{u}_s^c(\boldsymbol{k}), \; c k \hat{b}^c(\boldsymbol{k}) = \mu_0 \sum_s q_s n_s \hat{u}_s^c(\boldsymbol{k}).$$

Here $k_\parallel = \boldsymbol{k} \cdot \boldsymbol{B}_0 / B_0$. Solving ω we find $\boldsymbol{b} = \alpha_s \boldsymbol{u}_s$ with α_s indeed a constant, depending only on k (but not $k_\parallel$) which is fixed for the Beltrami mode. Since the dynamics of all species share

Examination and Unification: linear waves (L and R), nonlinear UNIchiral vortex

Two requirements in Meyrand-Galtier's theory to get |Pm|=1

described above. Now let us imagine that only $\mathbf{\Omega}_L$ vortices evolve: in this case, we must have $\mathbf{u}_R = 0$ which implies the condition $\mathbf{u} = d_I \nabla \times \mathbf{b}$. Under these conditions, Eq. (5) becomes:

$$\partial_t (1 - d_I^2 \Delta)\mathbf{b} = d_I \nabla \times [(\nabla \times \mathbf{b}) \times (1 - d_I^2 \Delta)\mathbf{b}]. \quad (6)$$

Linearising Eq. (6) about a static homogeneous magnetic field $\mathbf{B}_0$ gives $\omega_L (1 + d_I^2 k^2)\hat{\mathbf{b}} = d_I k_\| B_0 i \mathbf{k} \times \hat{\mathbf{b}}$, which yields in the limit $k d_I \gg 1$ nothing else than the dispersion relation of the left-handed circularly polarized cyclotron waves, i.e., $\omega_L = B_0 k_\| / (k d_I)$ [20]. How can we interpret

- one fluid: **u** and **b** aligned
- Linear wave with a background field: **b** and **k** x **b** aligned

$$\sigma_m = \frac{\hat{\mathbf{a}} \cdot \hat{\mathbf{b}}^* + \hat{\mathbf{a}}^* \cdot \hat{\mathbf{b}}}{2|\hat{\mathbf{a}}||\hat{\mathbf{b}}|}, \qquad \sigma_c = \frac{\hat{\mathbf{u}} \cdot \hat{\mathbf{b}}^* + \hat{\mathbf{u}}^* \cdot \hat{\mathbf{b}}}{2|\hat{\mathbf{u}}||\hat{\mathbf{b}}|}, \quad (3)$$

where ˆ means the Fourier transform, * the complex conjugate, and **a** the magnetic vector potential. From these quantities, we may define the magnetic polarization, $P_m = \sigma_m \sigma_c$, which varies by definition between -1 and $+1$.

magic? actually problematic

Let's check: $$\partial_t \boldsymbol{\Omega}_s = \nabla \times (\boldsymbol{v}_s \times \boldsymbol{\Omega}_s), \qquad (s = R, L)$$

where, $\boldsymbol{\Omega}_R = \boldsymbol{b}, \boldsymbol{v}_R = \boldsymbol{v} - \epsilon\nabla \times \boldsymbol{b}$ and $\boldsymbol{\Omega}_L = \boldsymbol{b} + \epsilon\nabla \times \boldsymbol{v}, \boldsymbol{v}_L = \boldsymbol{v}$.

Meyrand & Galtier (2012) also claimed that if we let the ion fluid speed $\boldsymbol{v}_L = \boldsymbol{v}$ be zero, the Hall MHD equation degenerates to the electron magnetohydrodynamics (EMHD)

$$\partial_t \boldsymbol{b} = -\epsilon\nabla \times [(\nabla \times \boldsymbol{b}) \times \boldsymbol{b}].$$

The electron speed satisfies the relation $\boldsymbol{v}_e = -\epsilon\nabla \times \boldsymbol{b}$. With a mean magnetic vector, the EMHD whistler wave dispersion relation $\omega = ck_{\|}k\epsilon B_0$ gives the electron speed $\boldsymbol{v}_e(\boldsymbol{k}) = -ck\epsilon\boldsymbol{b}(\boldsymbol{k})$ and magnetic vector potential $\boldsymbol{a}(\boldsymbol{k}) = \frac{i\hat{\boldsymbol{k}}\times\boldsymbol{b}(\boldsymbol{k})}{k} = \frac{c\boldsymbol{b}(\boldsymbol{k})}{k}$. In this case, P_m is of the indefinite form $0/0$, instead of being -1 as indicated by Meyrand & Galtier (2012) when interpreting the $k^{-7/3}$-sector spectrum as the result of EMHD; actually, $\partial_t \boldsymbol{b} = 0$ if $\boldsymbol{v} = 0$. So, it appears that the correct way

eMHD story has problem but that of iMHD can be ok

spectrum as the result of EMHD; actually, $\partial_t \boldsymbol{b} = 0$ if $\boldsymbol{v} = 0$. So, it appears that the correct way to reduce to EMHD is to "remove" $\boldsymbol{v}_L$ and its dynamics, instead of simply putting $\boldsymbol{v} = 0$; but, even if so, P_m is still undefined. Similarly, if we "remove" $\boldsymbol{v}_e$ and its dynamics, but not to simply let electron speed be zero as in Meyrand & Galtier (2012) (otherwise, $\partial_t \boldsymbol{b} = 0$), the Hall MHD equation degenerates to the ion magnetohydrodynamics (IMHD) equation

$$\partial_t (1 - \epsilon^2 \Delta) \boldsymbol{b} = \epsilon \nabla \times [(\nabla \times \boldsymbol{b}) \times (1 - \epsilon^2 \Delta) \boldsymbol{b}] \tag{18}$$

where ion speed $\boldsymbol{v}_i = \epsilon \nabla \times \boldsymbol{b}$. Then we linearize the IMHD equation about a mean magnetic field,

$$\omega (1 + k^2 \epsilon^2) \hat{\boldsymbol{b}}(\boldsymbol{k}) = k_{\parallel} \epsilon B_0 (i \boldsymbol{k} \times \hat{\boldsymbol{b}}(\boldsymbol{k})) \tag{19}$$

Remember:

- one fluid
- linear wave with a background field

So, let's turn to another treatment with helical representation

the ion fluid speed $\boldsymbol{v} = 0$, then the electron fluid speed $\boldsymbol{v}_e = -\epsilon\nabla \times \boldsymbol{b}$

Thus, Hall MHD equation degenerates to EMHD equation

$$\partial_t \boldsymbol{b} = -\epsilon\nabla \times [(\nabla \times \boldsymbol{b}) \times \boldsymbol{b}].$$

We define

$$\sigma_{ce} = \frac{\boldsymbol{v}_e(\boldsymbol{k}) \cdot \boldsymbol{b}^*(\boldsymbol{k}) + \boldsymbol{v}_e^*(\boldsymbol{k}) \cdot \boldsymbol{b}(\boldsymbol{k})}{2|\boldsymbol{v}_e(\boldsymbol{k})||\boldsymbol{b}(\boldsymbol{k})|}.$$

Due to $\boldsymbol{v}_e = -\epsilon\nabla \times \boldsymbol{b}$, $v_e^c(\boldsymbol{k}) = -ck\epsilon b^c(\boldsymbol{k})$ and

$$\sigma_{ce} = \frac{\sum_c -ck\epsilon|b^c(\boldsymbol{k})|^2}{\sum_c k\epsilon|b^c(\boldsymbol{k})|^2}.$$

if we restrict to only one chiral sector for every $\boldsymbol{k}$, $\sigma_m = c, \sigma_{ce} = -c$; $P_m = \sigma_m\sigma_{ce} = -c^2 = -1$. We should note that we didn't linearize the models, neither introduced the mean magnetic field.

similarly for iMHD

Since the spectra presented in Fig. 3 of Meyrand & Galtier (2012) are from those fluctuations of $P_m > 0.3$ and $P_m < -0.3$, their EMHD and IMHD interpretation thus should be combined with the argument of the dominance of one chiral sector at each k ($|P_m| = 1$ requires unichirality at each $\boldsymbol{k}$, but not necessarily homochirality for all $\boldsymbol{k}$.) Note that when Navier-Stokes equation

Insights from the fully two-fluid MHD and “electronic polarization”

$$\partial_t \boldsymbol{\Omega_s} = \nabla \times (\boldsymbol{v_s} \times \boldsymbol{\Omega_s}), \qquad (s = i, e)$$

assuming $\boldsymbol{v}_e = -\epsilon_e \nabla \times \boldsymbol{b}$, we have the EMHD equation:

$$\partial_t (1 - \epsilon_e^2 \Delta)\boldsymbol{b} = -\epsilon_e \nabla \times [(\nabla \times \boldsymbol{b}) \times (1 - \epsilon_e^2 \Delta)\boldsymbol{b}].$$

Here the EMHD equation includes electron inertia term. ϵ_e is the ratio between electron skin depth and normalization length scale. If we redefine the electron cross correlation as

$$\sigma_e = \frac{\hat{\boldsymbol{v}}_e \cdot \hat{\boldsymbol{b}}^* + \hat{\boldsymbol{u}}_e^* \cdot \hat{\boldsymbol{b}}}{2|\hat{\boldsymbol{u}}_e||\hat{\boldsymbol{b}}|},$$

using helical decomposition and choose only one chiral sector, we get $P_{me} = \sigma_m \sigma_e = -c^2 = -1$.

similarly for iMHD:

$$\sigma_i = \frac{\hat{\boldsymbol{v}}_i \cdot \hat{\boldsymbol{b}}^* + \hat{\boldsymbol{v}}_i^* \cdot \hat{\boldsymbol{b}}}{2|\hat{\boldsymbol{v}}_i||\hat{\boldsymbol{b}}|}$$

Thus, full two-fluid model clearly manifests that the really relevant "magnetic polarization" should actually be "electronic polarization", as the sign of P_m is determined by the charge, e or i.

Thank you!